International Journal of Theoretical Physics, Vol. 19, No. 7, 1980

Quantum Probability Zero-One Law for Sequential Terminal Events

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Received January 2, 1980

On the basis of the Jauch–Piron quantum probability calculus a zero–one law for sequential terminal events is proven, and the significance of certain crucial axioms in the quantum probability calculus is discussed. The result shows that the Jauch–Piron set of axioms is appropriate for the non-Boolean algebra of sequential events.

1. INTRODUCTION

In Rehder (1980), so-called *sequential connectives* $P \sqcap Q$, $P \sqcup Q$, $P \to Q$ (read: "P and then Q," "P or then Q," "if first P then Q") for projections P, Q in a complex Hilbert space H were derived from the "conditional probability operator" PQP in the following way:

$P \sqcap Q$ is the projection onto the orthocomplement of the null space $E_0(PQP)$ of PQP, i.e., $P \sqcap Q$ is the projection onto the closed range ("support") of PQP. The equality

$$\langle x, PQPx \rangle = \langle QPx, QPx \rangle = \| QPx \|^2$$

shows that $E_0(PQP) = E_0(QP)$, and we may write

$$P \sqcap Q = \{ x | QPx = 0 \}^{\perp}$$

Here, as always in the sequel, we identify projections with their ranges, e.g.,

$$P \sqcap Qx = x \Leftrightarrow x \in P \sqcap Q$$

It is easy to see that

$$P \sqcap Q = P \wedge (P^{\perp} \vee Q) \tag{1.1}$$

(cf. Rehder, 1980, Section 2).

Then $P \sqcup Q$ is defined as

$$\begin{aligned} P \sqcup Q &:= (P^{\perp} \sqcap Q^{\perp})^{\perp} \\ &= P \vee (P^{\perp} \wedge Q) \end{aligned} \tag{1.2}$$

and the material quasi-implication $P \rightarrow Q$ is given by

$$\begin{aligned} P \rightarrow Q &:= P^{\perp} \sqcup Q \\ &= P^{\perp} \vee (P \wedge Q) \end{aligned} \tag{1.3}$$

It was shown in Rehder (1980) that $\sqcap$ and $\sqcup$ are neither commutative nor associative; that $P \sqcap Q$ and $P \sqcup Q$ are not the infimum and supremum of P and Q, so that in particular, $\sqcap$, $\sqcup$ together with the inclusion $\leqslant$ ("implication") for subspaces in H do not define a lattice. (See, however, Kröger's study of so-called "Zwerch-Verbände" in Kröger (1973).

Instead, some new rules are valid, notably

$$P = Q \sqcap P + Q^{\perp} \sqcap P - I(P, Q)$$

where $I(P, Q)$ is interpreted as an interference term given by

$$P^{\perp} \sqcap (Q \sqcap P) = P^{\perp} \sqcap (Q^{\perp} \sqcap P)$$

(for this identity, see Rehder (1980). The meet and join of two projections can be written as sequential events, too:

$$P \wedge Q = P \sqcap (P^{\perp} \sqcup Q) = Q \sqcap (Q^{\perp} \sqcup P) \tag{1.4}$$

$$P \vee Q = P \sqcup (P^{\perp} \sqcap Q) = Q \sqcup (Q^{\perp} \sqcap P) \tag{1.5}$$

and for the classical material implication operator $P^{\perp} \vee Q$, we have

$$P^{\perp} \vee Q = P^{\perp} \sqcup (P \sqcap Q) \tag{1.6}$$

displaying a curious dual symmetry to formulas (1.1), (1.2), (1.3) above. Moreover, the well-known quantum-theoretical probability calculus (as in Jauch, 1968, p. 94) can be applied for sequential events (see Rehder, 1980, sec. 5).

In this paper we wish to show that with this probability theory, an analog to Kolmogorov's zero-one law, is true for (sequential) terminal σ-algebras (Theorem 3.6 below).

2. SEQUENTIAL σ- ALGEBRAS AND PROBABILITIES

Similar to classical probability theory, we call a family A of projections in H an *algebra*, if A is closed with respect to finite applications of $\sqcap$, $\sqcup$, $\rightarrow$, and orthocomplementation $^{\perp}$. The above identities (1.4), (1.5) show that A is also closed with respect to $\wedge$ and $\vee$.

Examples. A single projection P "generates" the algebra $\sigma(P):=\{0,H,P,P^{\perp}\}$. For two projections P and Q the "generated" algebra $\sigma(P,Q)$ has already infinitely many members, if P and Q do not commute: e.g., $P\sqcap Q$, $(P\sqcap Q)\sqcap Q$, $((P\sqcap Q)\sqcap Q)\sqcap Q$, etc.

As we mentioned in Section 1, associativity of three or more projections with respect to $\sqcap$ or $\sqcup$ is not generally valid, so that we have to use brackets in order to avoid ambiguities in terms containing repeated connectives. A simple convention is used, however, to make notations less clumsy: formulas are to be read from left to right, so that, e.g.,

$$P_1\sqcap P_2\sqcap P_3$$

means $(P_1\sqcap P_2)\sqcap P_3$, etc. Also let $\sqcap$ be stronger than $\wedge$, e.g., $P_1\wedge P_2\sqcap P_3=P_1\wedge(P_2\sqcap P_3)$. Deviating slightly from classical probability theory, we make the following definition.

2.1. *Definition.* An algebra A is a σ-*algebra* if from $P_n\in A$, $n=1,2,\ldots,P_n\rightarrow P$ weakly and P is a projection in H, it follows that $P\in A$.

Examples. The $\sqcup$-limit of a sequence of projections P_n, $n=1,2,\ldots,$ is defined as

$$\bigsqcup_{n=1}^{\infty} P_n:=P_1\sqcup P_2\sqcup P_3\sqcup\cdots$$

$\bigsqcup_{n=1}^{\infty}P_n$ is well defined and again a projection,

$$P_1\leqslant P_1\sqcup P_2\leqslant P_1\sqcup P_2\sqcup P_3\leqslant\cdots$$

so that

$$\bigsqcup_{n=1}^{\infty} P_n=P_1\vee(P_1\sqcup P_2)\vee(P_1\sqcup P_2\sqcup P_3)\vee\cdots$$

is an increasing sequence of projections having as its limit the closure of the union of $P_1,P_1\sqcup P_2,\ldots,P_1\sqcup P_2\sqcup\cdots\sqcup P_n$:

$$\bigsqcup_{n=1}^{\infty} P_n=\overline{\bigcup_{n=1}^{\infty}}(P_1\sqcup P_2\cdots\sqcup P_n)=\bigvee_{n=1}^{\infty}(P_1\sqcup P_2\cdots\sqcup P_n)$$

Analogously, the $\sqcap$-limit

$$\prod_{n=1}^{\infty} P_n := P_1 \sqcap P_2 \sqcap P_3 \sqcap \cdots$$

is a well-defined projection, which can be written as an intersection

$$\prod_{n=1}^{\infty} P_n = \bigcap_{n=1}^{\infty} (P_1 \sqcap P_2 \sqcap \cdots \sqcap P_n)$$

because $P_1 \geqslant P_1 \sqcap P_2 \geqslant P_1 \sqcap P_2 \sqcap P_3 \geqslant \cdots$ is a decreasing sequence of projections. The $\sqcup$-limit and the $\sqcap$-limit are weak limits of projections (Sz. Nagy, 1967, p. 17), so that they both belong to the σ-algebra generated by P_n, $n = 1, 2, \ldots$.

It is clear that for a family A_t, $t \in T$, of σ-algebras, the intersection $\cap_{t \in T} A_t$ is again a σ-algebra. $\sigma(P_i : i \in I)$ denotes the smallest σ-algebra containing all P_i, $i \in I : \sigma(P_i : i \in I)$ is said to be "generated" by P_i, $i \in I$. (cf. the above examples). For $I = \mathbb{N}$ we also write $\sigma(P_1, P_2, \ldots) = \sigma_1, \ldots, \sigma(P_n, P_{n+1}, \ldots) = \sigma_n$.

2.2. *Definition.* The terminal σ-algebra generated by a sequence of projections $P_1, P_2, \ldots$ is the σ-algebra

$$A_\infty := \bigcap_{n=1}^{\infty} \sigma_n$$

Following Jauch and Piron (Jauch, 1968, p. 94; Piron, 1976), we introduce a probability function w on a σ-algebra A. Let P, Q, P_n $(n \in \mathbb{N})$ be projections from a σ-algebra A.

2.3. *Definition.* A real-valued function w on A is called a normed additive functional iff

(W1) $\quad 0 \leqslant w(P) \leqslant 1$

(W2) $\quad w(0) = 0, \quad w(H) = 1$

(W3) $\quad PQ = 0 \Rightarrow w(P \sqcup Q) = w(P) + w(Q)$

(note that $P \sqcup Q = P + Q$).

Consequences.

(W4) $\quad w(P) = 1 - w(P^\perp), \quad$ from (W3), (W2)

(W5) $\quad P \leqslant Q \Rightarrow w(P) \leqslant w(Q)$

for

$$\begin{aligned} &P \leqslant Q \Rightarrow PQ^{\perp} = 0 \\ &\quad \Rightarrow w(P) + w(Q^{\perp}) = w(P \sqcup Q^{\perp}) \qquad \text{from (W3)} \\ &\quad \Rightarrow w(P) = w(P \sqcup Q^{\perp}) - w(Q^{\perp}) \\ &\qquad\qquad = w(P \sqcup Q^{\perp}) - 1 + w(Q) \qquad \text{from (W4)} \\ &\qquad\qquad \leqslant w(Q) \qquad \text{from (W1)} \end{aligned}$$

(W6) $\quad w(P \sqcup Q) \leqslant w(P) + w(Q)$

for

$$\begin{aligned} w(P \sqcup Q) &= w(P) + w(P^{\perp} \wedge Q) \qquad \text{from (W3)} \\ &\leqslant w(P) + w(Q) \qquad \text{from (W5)} \end{aligned}$$

(W7) $\quad w\left(\bigsqcup_{n=1}^{m} P_n \right) \leqslant \sum_{n=1}^{m} w(P_n) \qquad$ by induction from (W6)

(W8) $\quad w\left(\bigsqcup_{n=1}^{\infty} P_n \right) \leqslant \sum_{n=1}^{\infty} w(P_n)$

by taking limits in (W7). Observe that $\bigsqcup_{n=1}^{\infty} P_n \in A$ and $w(\bigsqcup_{n=1}^{\infty} P_n) \leqslant 1$, but $\Sigma_{n=1}^{\infty} w(P_n)$ need not be finite.

2.4. *Definition*. A normed additive functional w on A is called a probability function on A iff

(W9) $\quad w$ is continuous with respect to weak limits in A: if P_n and P are projections in A and $P_n \to P$ weakly then $w(P_n) \to w(P)$.

(W10) $\quad w(P) = 1,\ w(Q) = 1 \Rightarrow w(P \wedge Q) = 1$

(W9) implies, in particular, that

$$w\left(\bigsqcup_{n=1}^{\infty} P_n \right) = \lim_{n\to\infty} w(P_1 \sqcup P_2 \sqcup \cdots \sqcup P_n)$$

$$w\left(\bigsqcap_{n=1}^{\infty} P_n \right) = \lim_{n\to\infty} w(P_1 \sqcap P_2 \sqcap \cdots \sqcap P_n)$$

For a very special case, axiom (W10) provides finite (sub-) additivity: (W10) is equivalent to

$$w(P) = 0, \qquad w(Q) = 0 \Rightarrow w(P \vee Q) = 0$$

that is

$$w(P \vee Q) = w(P) + w(Q)$$

similarily (by induction) for m "impossible" events:

$$\text{(W11)} \qquad w(P_1) = \cdots = w(P_m) = 0 \Rightarrow w\left(\bigvee_{j=1}^{m} P_i \right) = 0.$$

(W10) is perhaps the most controversial of the axioms defining a quantum theoretical probability function, even though it can be shown to be always true in classical (Kolmogorov) probability theory on Boolean lattices (see Jauch, 1976, p. 136).

Jauch has also given simple physical interpretations for (W10) via filters. (W10) enters into the proof about the impossibility of hidden variables (Jauch, 1968, p. 117: Proof of the lemma; Jauch and Piron, 1963). This proof has been challenged by Bell (1966) and Bub (1974, pp. 60–64) by a criticism of (W10).

We do not require σ-additivity for w:

$$P_n P_m = 0, \qquad n \neq m \Rightarrow w\left(\bigsqcup_{n=1}^{\infty} P_n \right) = \sum_{n=1}^{\infty} w(P_n)$$

(See, however, Jauch, 1968, p. 97, problem 9; and Neveu, 1964, p. 11.) In case of σ-additivity, w can be represented by an operator ρ of trace class:

$$w(P) = \operatorname{Tr} \rho P$$

For the purpose of the present paper, the above axioms (W1), (W2), (W3), (W9), and (W10) are sufficient. [(W10) will only be used in (4.2).] It is important, however, that all trace-class probability functions also fulfill (W9):

2.5. Theorem. If a sequence of projections P_n converges weakly towards a projection P, then

$$\operatorname{Tr} \rho P_n \rightarrow \operatorname{Tr} \rho P$$

where ρ is a positive trace class operator with trace 1.

Proof. Cf. Jauch (1976), p. 137–138: Jauch's proof carries over here for weak convergence as well, since we assume uniform boundedness of P (which in Jauch's proof is deduced from strong convergence). ■

2.6. *Lemma*. If $P_1, P_2, \ldots, P_n$ commute, then

$$w(P_1 \vee P_2 \cdots P_n) \leqslant \sum_{m=1}^{n} w(P_m)$$

Proof. $n=2$

$$\begin{aligned} w(P_1 \vee P_2) &= w(P_1 + P_1^\perp \wedge P_2) \\ &= w(P_1) + w(P_1^\perp \wedge P_2) \qquad \text{from (W3)} \\ &\leqslant w(P_1) + w(P_2) \qquad \text{from (W5)} \end{aligned}$$

The general case follows by induction. ■

It is remarkable that $w(P_1 \vee P_2) \leqslant w(P_1) + w(P_2)$ cannot be true for noncommuting P_1, P_2 and for *every* normed additive functional w:

$$P_1 \vee P_2 = P \sqcup (P_1^\perp \sqcap P_2)$$

implies

$$w(P_1 \vee P_2) = w(P_1) + w(P_1^\perp \sqcap P_2)$$

Now $w(P_1^\perp \sqcap P_2) \leqslant w(P_2)$ for every w implies

$$\langle \varphi, P_1^\perp \sqcap P_2 \varphi \rangle \leqslant \langle \varphi, P_2 \varphi \rangle$$

for every state $\varphi \in H$, and this means

$$P_1^\perp \sqcap P_2 \leqslant P_2$$

i.e., P_1 and P_2 commute! If, however, P_1 and P_2 do not necessarily commute but are independent with respect to a fixed probability function w, we have again finite subadditivity:

$$\begin{aligned} w(P_1 \vee P_2) &= w(P_1) + w(P_1^\perp \sqcap P_2) \\ &= w(P_1) + w(P_1^\perp) w(P_2) \qquad \text{(see Definition 2.7 below)} \\ &\leqslant w(P_1) + w(P_2) \qquad \text{from (W1)} \end{aligned}$$

2.7. *Definition*. Let $C_1, C_2, \ldots$ be a sequence of algebras of projections in H, and let a probability function w be defined on all C_n. The sequence (C_n) is called *sequentially w-independent*, if for every finite choice of algebras

$$C_{n_1}, C_{n_2}, \ldots, C_{n_k}$$

and any $P_{n_i} \in C_{n_i}$, $i := 1, \ldots, k$, the probability $w(P_{n_1} \sqcap \cdots \sqcap P_{n_k})$ does not depend upon the order of the P_{n_i}, $i=1,\ldots,k$, and is equal to the product $\Pi_{i=1}^{k} w(P_{n_i})$.

In order to get a better understanding of w independence, let $w=w_\varphi$ be given by a fixed state vector $\varphi \in H$:

$$w_\varphi(P)=\langle \varphi, P\varphi \rangle$$

for projections P. Let us discuss w_φ independence of two projections P, Q. Assume that P and Q commute with respect to φ:

$$PQ\varphi = QP\varphi$$

It follows

$$PQ^{\perp}\varphi = Q^{\perp}P\varphi \tag{2.1}$$

Then

$$\langle \varphi, P \wedge Q\varphi \rangle = \lim_n \langle \varphi, (PQ)^n \varphi \rangle = \langle \varphi, PQ\varphi \rangle$$

and observing

$$\begin{aligned} \langle \varphi, P \sqcap Q\varphi \rangle &= \left\langle \varphi, \left(P - \lim_k (PQ^{\perp})^k \right)\varphi \right\rangle \\ &= \langle \varphi, P\varphi \rangle - \lim_k \langle \varphi, (PQ^{\perp})^k \varphi \rangle \end{aligned}$$

together with equation (2.1), yields

$$\begin{aligned} \langle \varphi, P \sqcap Q\varphi \rangle &= \langle \varphi, P\varphi \rangle - \langle \varphi, PQ^{\perp}\varphi \rangle \\ &= \langle \varphi, (P - PQ^{\perp})\varphi \rangle = \langle \varphi, PQ\varphi \rangle \\ &= \langle \varphi, Q \sqcap P\varphi \rangle \end{aligned}$$

Now we can easily prove the following theorem.

2.8. *Theorem.* Assume that P and Q commute with respect to φ:

$$PQ\varphi = QP\varphi$$

Then P and Q are w_φ independent if and only if $(Q - w_\varphi(Q))\varphi$ and $(P - w_\varphi(P))\varphi$ are orthogonal with respect to the inner product $\langle\ ,\ \rangle$.

Proof.

$$\begin{aligned}
&\langle (Q - w_\varphi(Q))\varphi, (P - w_\varphi(P))\varphi \rangle \\
&\quad = \langle \varphi, (Q - w_\varphi(Q))\,(P - w_\varphi(P))\varphi \rangle \\
&\quad = \langle \varphi, QP\varphi \rangle - w_\varphi(P)\langle \varphi, Q\varphi \rangle - w_\varphi(Q)\langle \varphi, P\varphi \rangle + w_\varphi(P)w_\varphi(Q) \\
&\quad = \langle \varphi, QP\varphi \rangle - w_\varphi(P)w_\varphi(Q)
\end{aligned}$$

As we have shown above

$$\langle \varphi, QP\varphi \rangle = \langle \varphi, P \sqcap Q\varphi \rangle$$

and this proves the theorem. ■

As the probability w will be fixed from now on, we shall just say *independent* instead of *sequentially w-independent*.

3. A ZERO–ONE LAW FOR SEQUENTIAL EVENTS

We are now ready to prove that for every event T in the sequential terminal σ-algebra A_∞ either $w(T) = 0$ or $w(T) = 1$. Imitating as far as possible the procedure used for the corresponding proof in classical probability theory (cf. Chung, 1968, and Neveu, 1964), we introduce the concept of limit classes.

3.1. *Definition.* A family L of projections in H is called a limit class if from $P_n \in L$, $n = 1, 2, \ldots, P_n \to P$ weakly, where P is a projection, it follows that $P \in L$. By $\lambda(C)$ we denote the smallest limit class containing all projections from the family C; $\lambda(C)$ is "generated" by C.

It is clear that every σ-algebra is a limit class. More precisely, we have the following lemma.

3.2. *Lemma.* Let A be an algebra of projections in H. Then A is a σ-algebra $\Leftrightarrow$ A is a limit class.

The next lemma is trickier. It will be essential in the proof of our zero–one law.

3.3. *Lemma.* If A is an algebra, then $\lambda(A) = \sigma(A)$.

Proof. (a) We prove first that for every $Q \in \lambda(A)$, $C_Q := \{P \mid P \sqcup Q, Q \sqcup P, P \sqcap Q^\perp, Q \sqcap P^\perp \in \lambda(A)\}$ is a limit class. Let $P_n \in C_Q$, and $P_n \to P$ weakly for a projection P. Since $P_n \sqcup Q = E_0(Q^\perp P_n^\perp)$, and $Q^\perp P_n^\perp \to$

$Q^{\perp}P^{\perp}$ weakly, $P \sqcup Q \in C_Q$ because $\lambda(A)$ is a limit class. The same argument applies for $Q \sqcup P$, $P \sqcap Q^{\perp}$, $Q \sqcap P^{\perp}$.

(b) Secondly, we observe that for all projections $R \in A$, $A \subset C_R$, because A is an algebra, and $A \subset \lambda(A)$. It follows that for all $R \in A$, $\lambda(A) \subset C_R$, because C_R is a limit class. This means for every $P \in \lambda(A): P \sqcup R$, $R \sqcup P$, $P \sqcap R^{\perp}$, $R \sqcap P^{\perp} \in \lambda(A)$; hence $A \subset C_p$ for every $P \in \lambda(A)$ and thus $\lambda(A) \subset C_P$. Choosing $R=0$ shows that $\lambda(A)$ is closed with respect to orthocomplements, so that $\lambda(A)$ is an algebra. ■

It follows from Lemmas 3.3 and 3.2 that $\lambda(A)$ is even a σ-algebra:

3.4. *Corollary*. If A is an algebra, then $\lambda(A) = \sigma(A)$.

The following lemma ensures that independence of a σ-algebra A and an algebra C is inherited by A and $\sigma(C)$.

3.5. *Lemma*. Let A be a σ-algebra and C be an algebra of projections in H. If A and C are independent, then A and $\sigma(C)$ are also independent.

Proof. (a) We prove that the family

$$D = \{P \mid P \text{ independent of } A\}$$

is a limit class. Let P_n and P be projections, $P_n \in D$, $n=1,2,\ldots$, and $P_n \to P$ weakly. It follows that, for $R \in A$, $R \sqcap P_n \to R \sqcap P$ and $P_n \sqcap R \to P \sqcap R$ weakly. Continuity of w [see (W3)] yields independence of P and R.

(b) Obviously $C \subset D$, so that together with Corollary 3.4: $\sigma(C) = \lambda(C) \subset D$, which had to be proved. ■

3.6. *Theorem*. Let (P_n) be an independent sequence of projections in H. For every $T \in A_\infty$, either $w(T)=0$ or $w(T)=1$.

Proof. We prove $w(T \sqcap T) = w(T)^2$. This, and $T \sqcap T = T$, proves the theorem. First, we observe that the two algebras $\sigma(P_1,\ldots,P_m)$ and $\sigma(P_{m+1},P_{m+1},\ldots) = \sigma_{m+1}$ are independent for every $m \in \mathbb{N}$. As $A_\infty \subset \sigma_{m+1}$ for every m, A_∞ and $\sigma(P_1,\ldots,P_m)$ are independent for every m, so that A_∞ is also independent of the algebra $\cup_{m=1}^{\infty}\sigma(P_1,\ldots,P_m) =: C$ so that Lemma 3.5 applies: A_∞ and $\sigma(C)$ are independent. On the other hand,

$$A_\infty \subset \sigma_1 = \sigma(P_1,P_2,\ldots) \subset \sigma(C) = \sigma\left(\bigcup_{m=1}^{\infty} \sigma(P_1,\ldots,P_m)\right)$$

which means that A_∞ must be independent of itself, i.e.,

$$w(T \sqcap T) = w(T) \cdot w(T)$$

for every $T \in A_\infty$. ■

We give an example of two terminal events in the sequential terminal σ-algebra

$$A_\infty = \bigcap_{n=1}^{\infty} \sigma_n$$

where $\sigma_n = \sigma(P_n, P_{n+1}, \ldots)$. Choose a sequence of projections $E_n \in \sigma_n$, and let

$$\begin{aligned} E_n^+ &:= E_n \sqcup E_{n+1} \sqcup \cdots \\ &= \bigcup_{m=n}^{\overline{\infty}} (E_n \sqcup E_{n+1} \sqcup \cdots \sqcup E_m) = \bigvee_{m=n}^{\infty} (E_n \sqcup E_{n+1} \sqcup \cdots \sqcup E_m) \end{aligned}$$

Obviously, $E_n^+ \in \sigma_n$, and thus

$$E_\infty := \bigcap_{n=1}^{\infty} E_n^+ \in \bigcap_{n=1}^{\infty} \sigma_n = A_\infty$$

$x \in E$ iff $x \in E_n^+$ for all $n \in \mathbb{N}$, i.e., x belongs to every "tail"

$$E_n \sqcup E_{n+1} \sqcup \cdots$$

which means that for every "moment" n, "E_n or then E_{n+1} or then $\cdots$ ad inf."

Now define

$$\begin{aligned} F_n^+ &:= E_n \sqcap E_{n+1} \sqcap \cdots \\ &= \bigcap_{m=n}^{\infty} (E_n \sqcap E_{n+1} \sqcap \cdots \sqcap E_m) \in \sigma_n \end{aligned}$$

and

$$F_\infty := \bigvee_{n=1}^{\infty} \bigcap_{m=n}^{\infty} (E_n \sqcap E_{n+1} \sqcap \cdots \sqcap E_m) \in A_\infty$$

Every $x \in F_\infty$ is the limit of linear combinations of vectors $y \in F_n^+$, and for these y we have $y \in (E_n$ and then E_{n+1} and then $E_{n+2} \ldots$ ad inf.), or $y \in E_n$ and $y \in E_n \sqcap E_{n+1}$ and $Y \in e_n \sqcap E_{n+1} \sqcap E_{n+2}$ and…, for a certain n.

Remark. In classical probability theory, where all projections E_n commute, E_∞ is the event "E_n is observed at infinitely many instants n", or, shorter: "E_n infinitely often". F_∞ represents "From a certain time n_0 on, all E_n, $n \geqslant n_0$, happen", or shorter: "E_n almost always."

4. A BOREL–CANTELLI CRITERION FOR SEQUENTIAL EVENTS

Theorem 3.6 states that E_∞ is either impossible: $w(E_\infty)=0$, or certain: $w(E_\infty)=1$. There is a Borel–Cantelli-type criterion to decide which of the two actually occurs.

4.1. *Theorem*. For projections E_n in H

$$\sum_{n=1}^{\infty} w(E_n)<\infty \Rightarrow w(E_\infty)=0$$

Proof.

$$w(E_\infty)=w\left(\bigcap_{n=1}^{\infty} E_n^+\right)=w\left(\lim_{m\to\infty}\bigcap_{n=1}^{m} E_n^+\right)$$

$$=\lim_{m\to\infty} w\left(\bigcap_{n=1}^{m} E_n^+\right) \qquad \text{from (W9)}$$

$$\leqslant \lim_{m\to\infty} w(E_m^+) \qquad \text{from (W5)}$$

$$\leqslant \lim_{m\to\infty} \sum_{j=m}^{\infty} w(E_j)=0 \qquad \text{from (W7)}$$

■

There is a striking converse to Theorem 4.1 for independent projections E_n:

4.2. *Theorem*. If the sequence (E_n) of projections in H is independent, then

$$\sum_{n=1}^{\infty} w(E_n)=\infty \Rightarrow w(E_\infty)=1$$

Proof.

$$w(E_\infty^\perp)=w\left(\bigvee_{n=1}^{\infty}(E_n^\perp\sqcap E_{n+1}^\perp\sqcap\cdots)\right)=w\left(\lim_{m\to\infty}\bigvee_{n=1}^{m}(E_n^\perp\sqcap E_{n+1}^\perp\sqcap\cdots)\right)$$

$$=\lim_{m\to\infty} w\left(\bigvee_{n=1}^{m}(E_n^\perp\sqcap E_{n+1}^\perp\sqcap\cdots)\right) \qquad \text{from (W9)}$$

but

$$w(E_n^{\perp}\sqcap E_{n+1}^{\perp}\sqcap\cdots)=w\left(\bigcap_{m=n}^{\infty}(E_n^{\perp}\sqcap\cdots\sqcap E_m^{\perp})\right)$$

$$=w\left(\lim_{k\to\infty}\bigcap_{m=n}^{k}(E_n^{\perp}\sqcap\cdots\sqcap E_m^{\perp})\right)\qquad\text{(W9)}$$

$$\leqslant\lim_{k\to\infty}w(E_n^{\perp}\sqcap\cdots\sqcap E_k^{\perp})\qquad\text{(W5)}$$

$$=\lim_{k\to\infty}\prod_{j=n}^{k}\left[1-w(E_j)\right]\qquad(2.7)$$

$$\leqslant\lim_{k\to\infty}\exp\left[-\sum_{j=n}^{k}w(E_j)\right]=0$$

(W11) yields $w(E_{\infty}^{\perp})=0$, i.e., $w(E_{\infty})=1$, by (W4). ■

If the σ-algebras σ_n are generated by independent projections P_j, then Theorem 4.2 can be applied to $E_j=P_j$.

We conclude this paper with a remark on the Jauch–Piron axiom (W10). The axiom

(W10) $\quad w(P)=1,\quad w(Q)=1\Rightarrow w(P\wedge Q)=1$

or equivalently

(W11) $\quad w(P)=0,\quad w(Q)=0\Rightarrow w(P\vee Q)=0$

entered the proof of the Borel–Cantelli Theorem 4.2 at a crucial point. (W10) and (W11) are properties of w, holding for all projections P, Q in H. If, on the other hand, we fix a pair of projections P, Q and assume (W11) to be true for all normed additive functionals w, then P and Q must commute!

It suffices to show that for the interference term $I(P,Q)$

$$w_{\varphi}(I(P,Q))=0\qquad(7.1)$$

holds for all φ from a complete orthonormal system of vectors φ in H.

From $w_{\varphi}(P)=w_{\varphi}(Q)=0$ and $0=w_{\varphi}(P\vee Q)=w_{\varphi}(Q)+w_{\varphi}(Q^{\perp}\sqcap P)$ follows $w_{\varphi}(Q^{\perp}\sqcap P)=0$; and from $Q\sqcap P\leqslant Q$ we know also that $w_{\varphi}(Q\sqcap P)=0$ [by (W5) and (W1)].

The identity (cf. Section 1)

$$P=Q\sqcap P+Q^{\perp}\sqcap P-I(P,Q)$$

and linearity of w_{φ} prove (7.1).

The following suggestive reformulation is perhaps more intuitive:

If from the certainty (truth) of P and Q for all preparations φ it follows that $P \wedge Q$ is also certain (true), then P and Q must be commensurable.

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